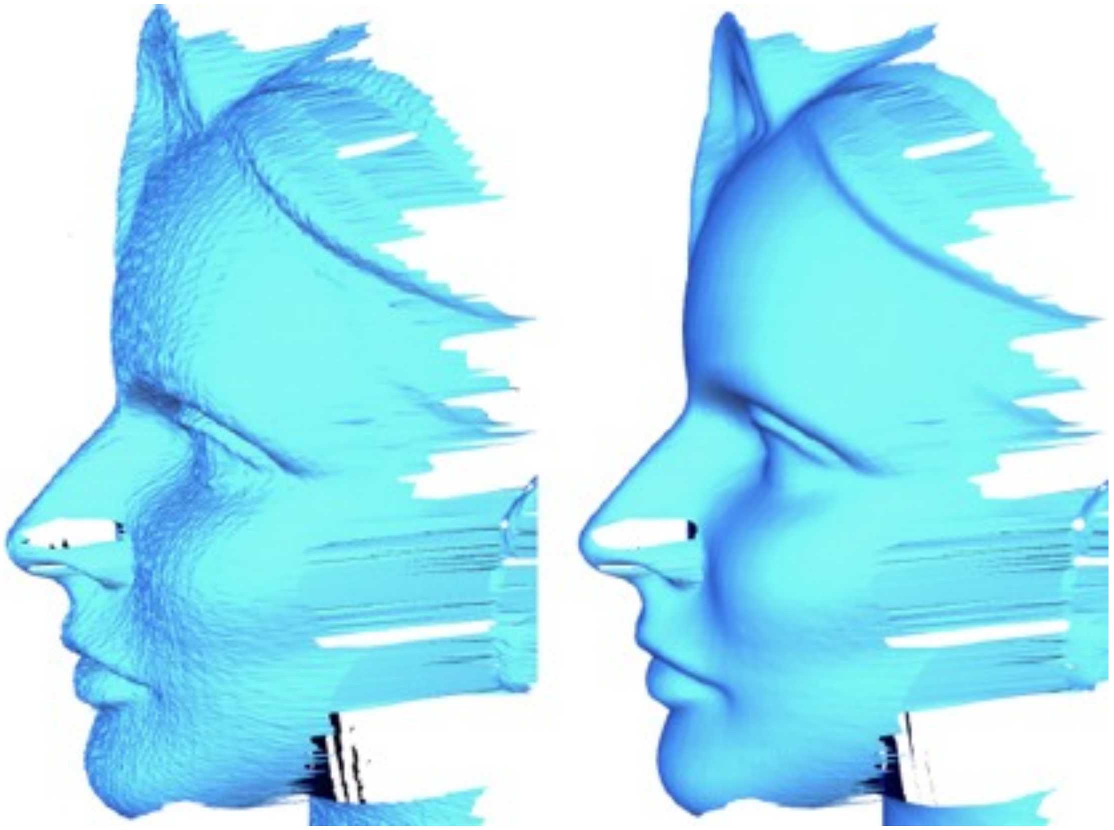
symposium on geometry processing
SGP 2011

Smoothing & Fairing

Mario Botsch

Motivation

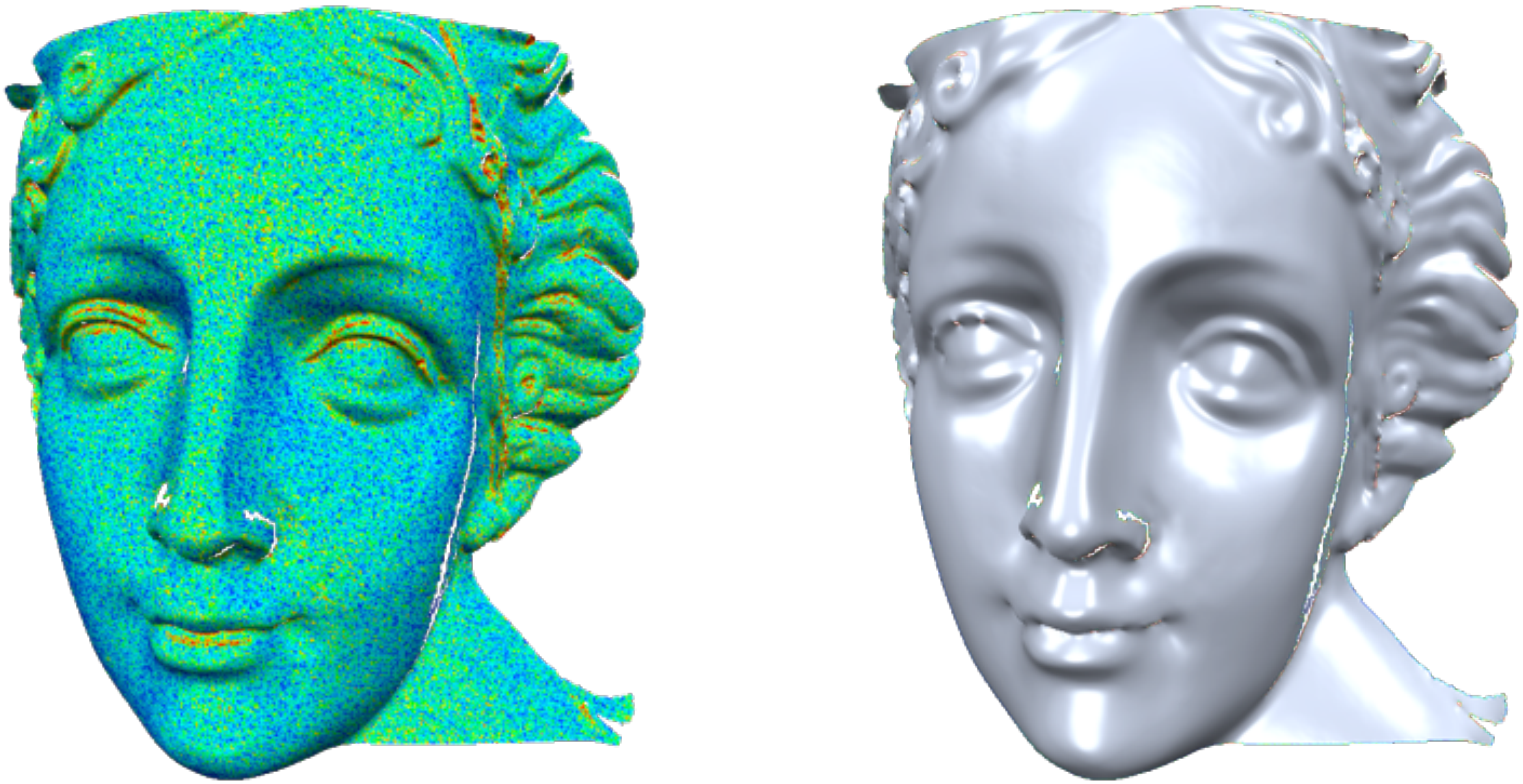
- Filter out high frequency noise



Desbrun, Meyer, Schroeder, Barr: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 99

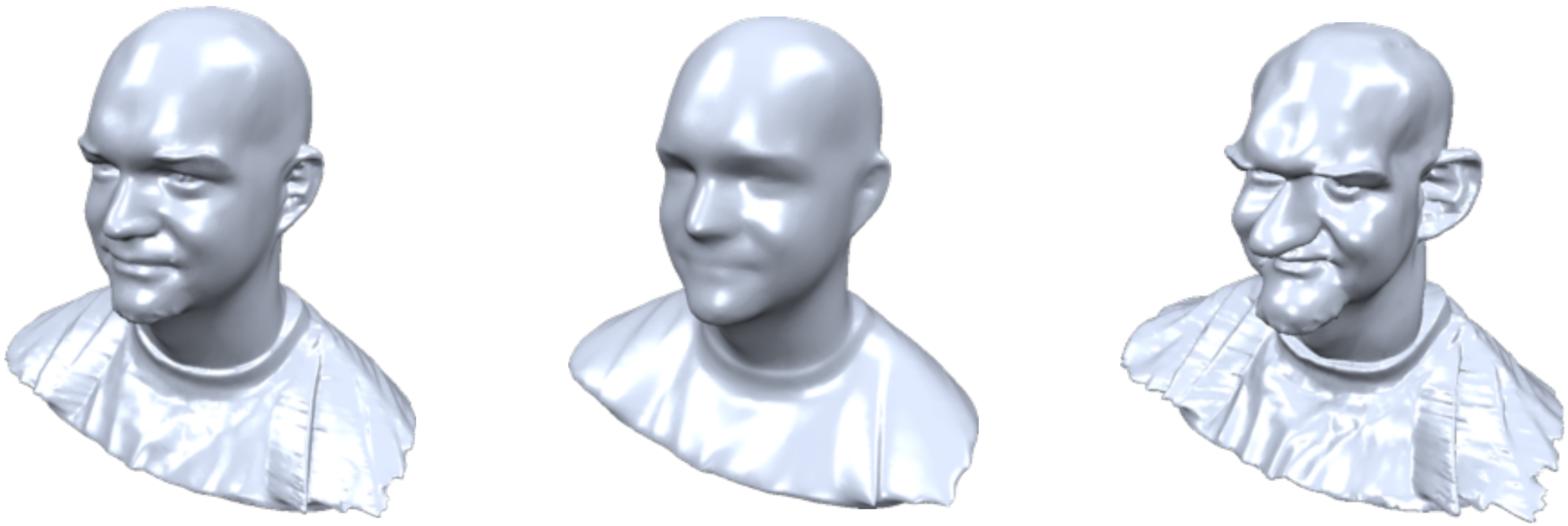
Motivation

- Filter out high frequency noise



Motivation

- Advanced Filtering

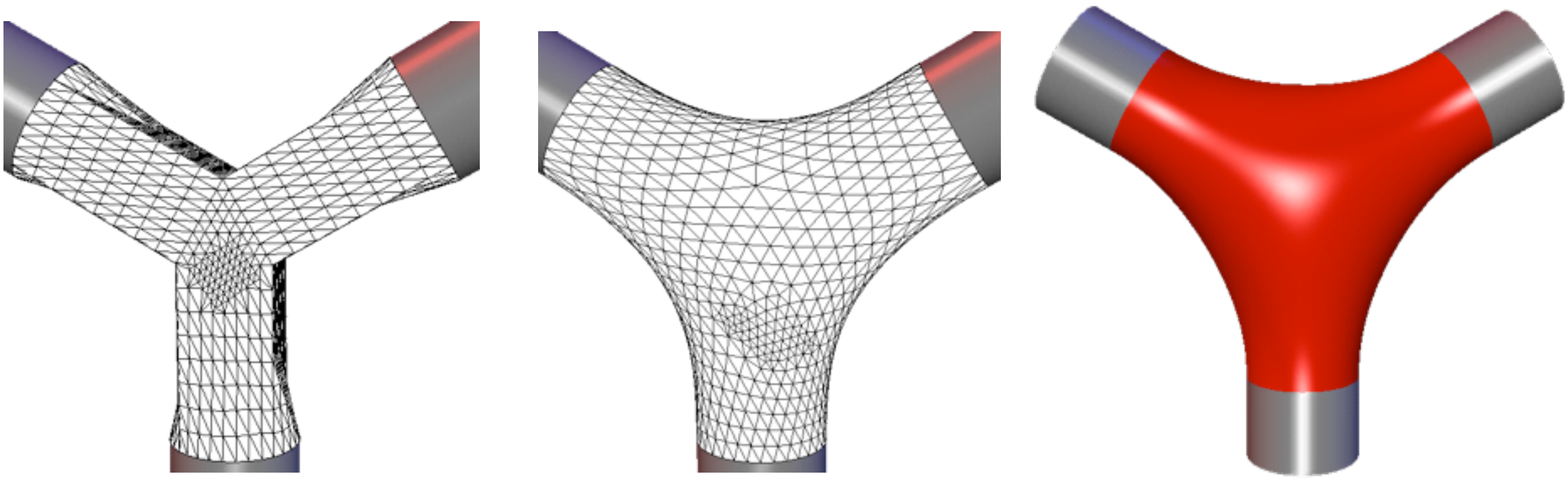


Original　　　Low-Pass　　　Exaggerate

Kim, Rosignac: *Geofilter: Geometric Selection of Mesh Filter Parameters*, Eurographics 05

Motivation

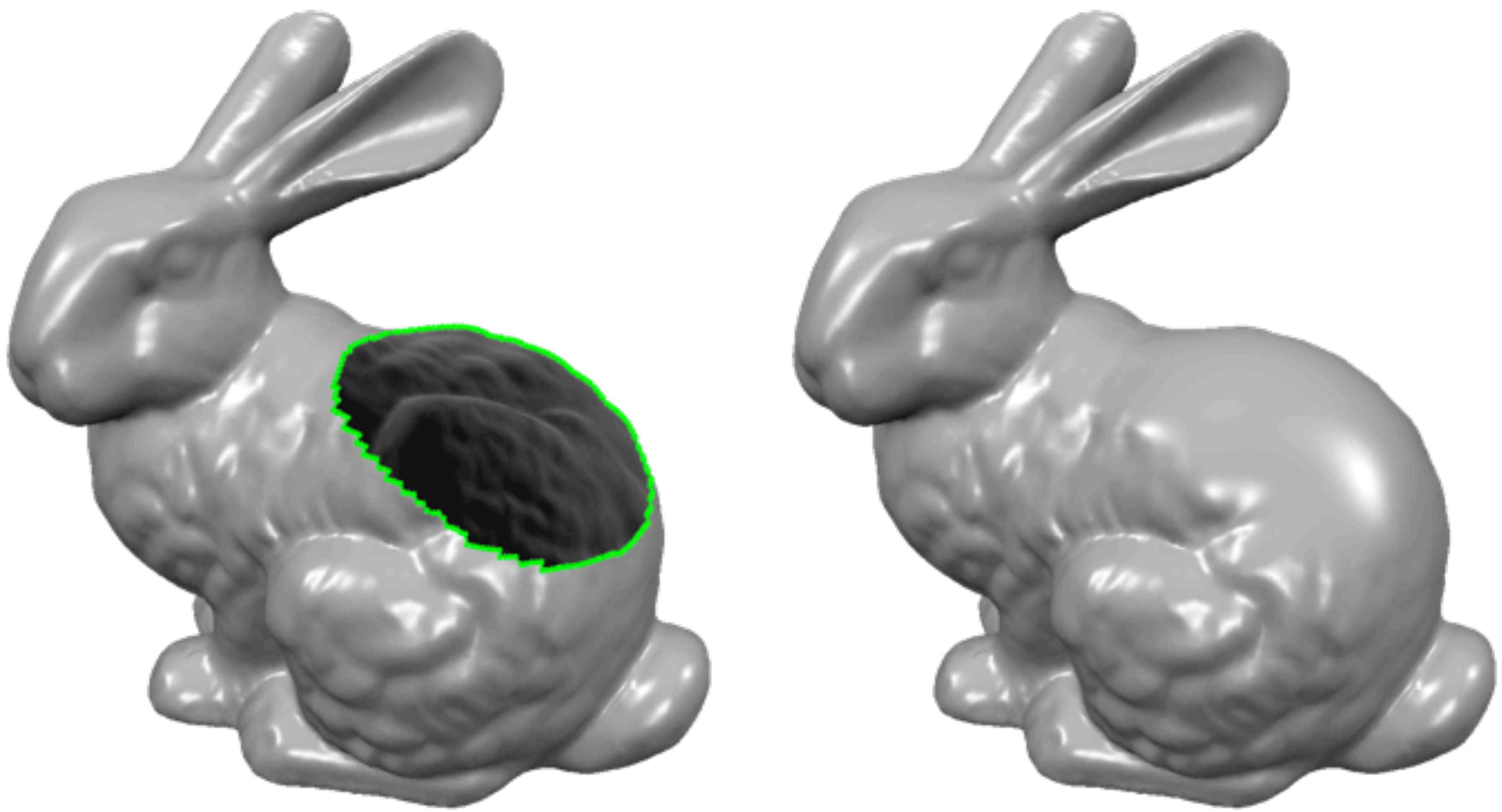
- Fair Surface Design



Schneider, Kobbelt: *Geometric fairing of irregular meshes for free-form surface design*, CAGD 18(4), 2001

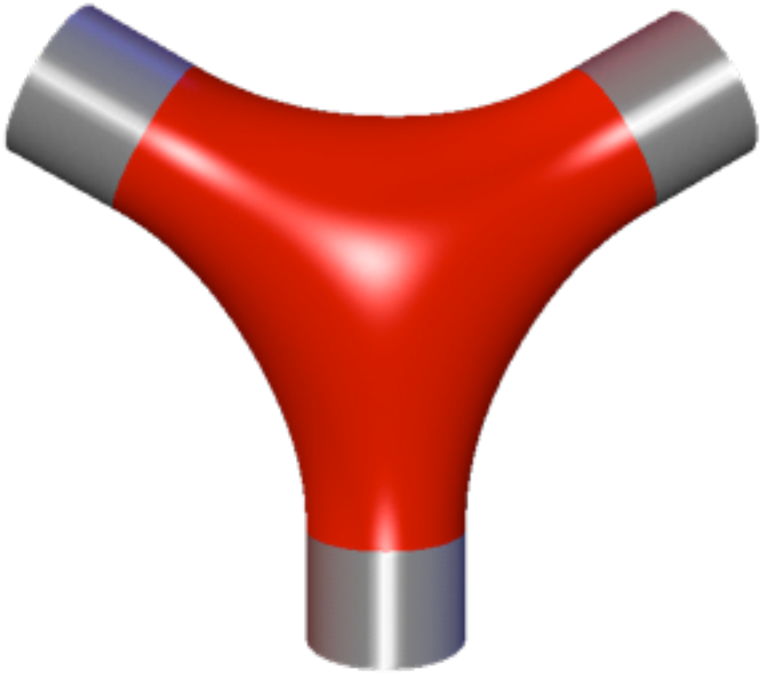
Motivation

- Hole-filling with energy-minimizing patches

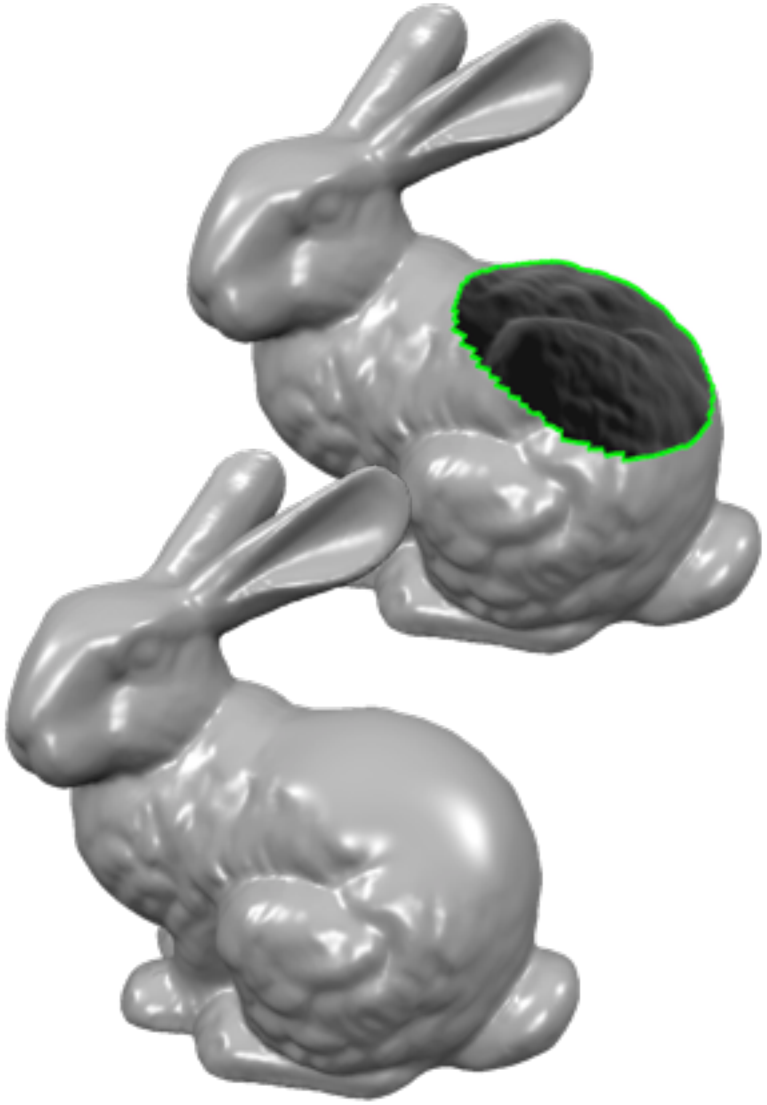


Liepa: *Filling Holes in Meshses,* SGP 2003

Motivation



Demo



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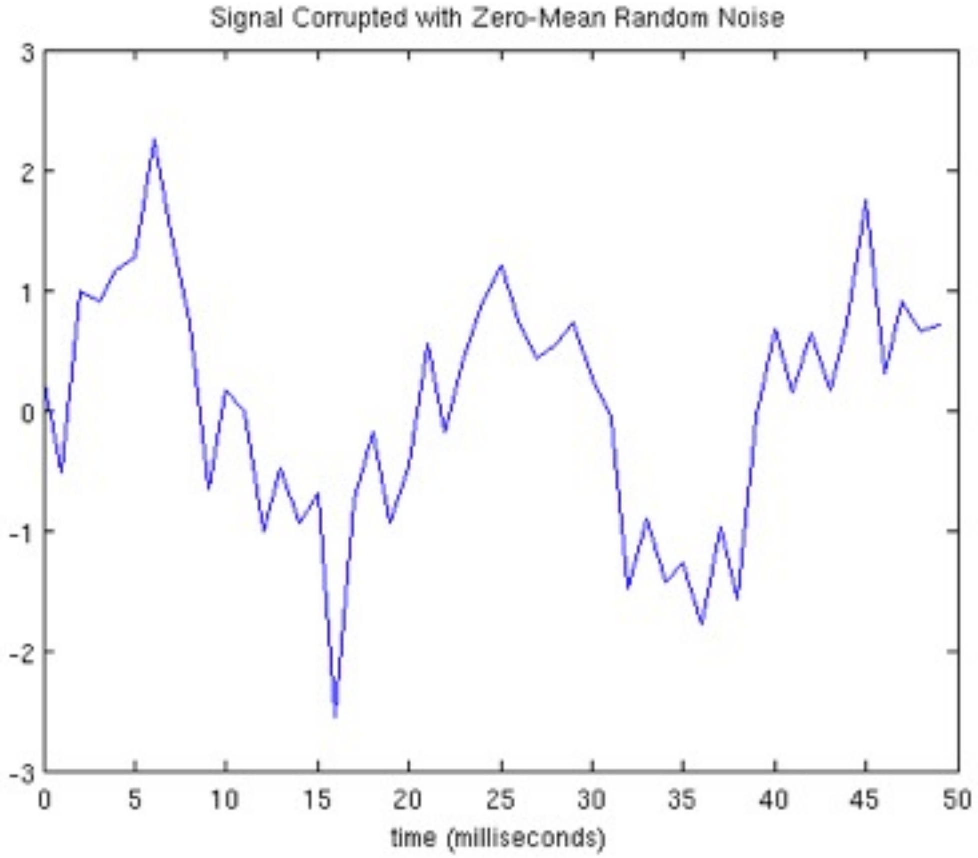
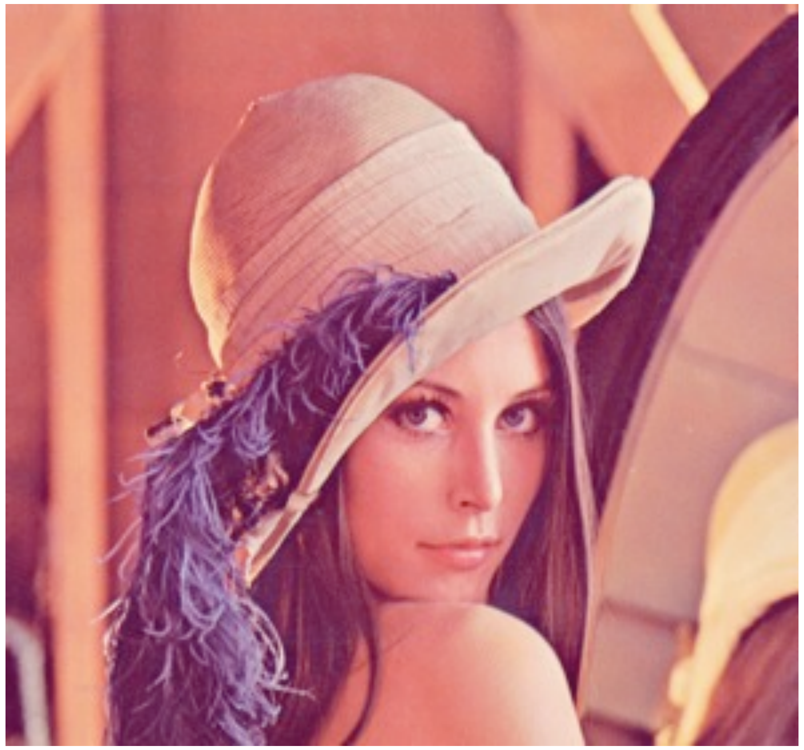
<u>Smoothing</u> & Fairing

Smoothing

- How would you smooth a 1D/2D signal?





Diffusion Flow

diffusion constant

$$\frac{\partial f}{\partial t} = \lambda \Delta f$$

Laplace operator

Diffusion Flow

- 2nd order elliptic PDE

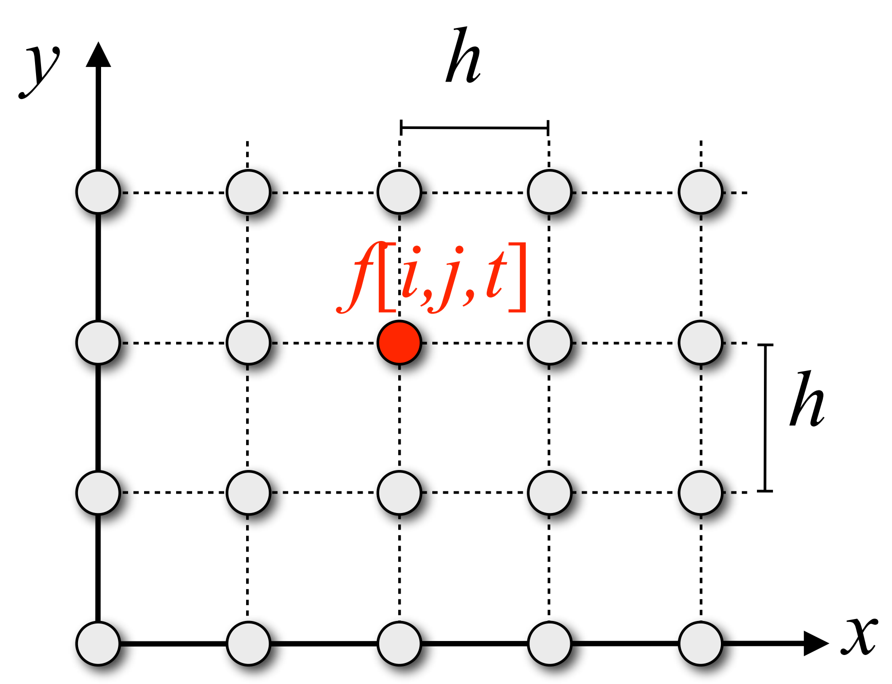
$$\frac{\partial f(x,y,t)}{\partial t} = \lambda \left(\frac{\partial^2 f(x,y,t)}{\partial x^2} + \frac{\partial^2 f(x,y,t)}{\partial y^2} \right)$$

- Solve numerically
  - Discretize in space & time
  - Discretize time derivative
  - Discretize spatial derivatives

Discretize in Space & Time

- Sample function $f(x,y,t)$ on a regular grid
  - Grid spacing h, time step δt

$$f[i,j,t] \;=\; f\left(i \cdot h, j \cdot h, t \cdot \delta t\right), \qquad \begin{aligned} i &= 1, \ldots, n, \\ j &= 1, \ldots, m, \\ t &= 0, 1, 2, \ldots \end{aligned}$$



Finite Differences

Brook Taylor
(1685-1731)

- Approximate $f(x+h)$ from Taylor series

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots \\ &\approx f(x) + hf'(x) \end{aligned}$$

- Approximate of $f'(x)$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Finite Differences

- Approximate $f(x\text{-}h)$ from Taylor series

$$\begin{aligned} f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) + \ldots \\ &\approx f(x) - hf'(x) \end{aligned}$$

Brook Taylor
(1685-1731)

- Approximate of $f'(x)$

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

Finite Difference Method

- Approximation of spatial derivatives

$$f_x[i,j,t] \approx \frac{f[i+1,j,t] - f[i,j,t]}{h}$$

forward differences

$$f_x[i,j,t] \approx \frac{f[i,j,t] - f[i-1,j,t]}{h}$$

backward differences

$$f_x[i,j,t] \approx \frac{f[i+1,j,t] - f[i-1,j,t]}{2h}$$

central differences

Finite Difference Method

- Approximation of higher-order derivatives

$$f_{xx}[i,j,t] \approx \frac{f_x[i,j,t] - f_x[i-1,j,t]}{h}$$

$$= \frac{f[i+1,j,t] - 2f[i,j,t] + f[i-1,j,t]}{h^2}$$

- Approximation of Laplacian

$$\Delta f[i,j,t] \approx f_{xx}[i,j,t] + f_{yy}[i,j,t]$$

$$= \frac{f[i+1,j,t] + f[i-1,j,t] + f[i,j+1,t] + f[i,j-1,t] - 4f[i,j,t]}{h^2}$$

Finite Differences

- Approximate $f(t+\delta t)$ from Taylor series

$$
\begin{aligned}
f(t+\delta t) &= f(t) + \delta t\, \dot{f}(t) + \frac{\delta t^2}{2!} \ddot{f}(t) + \ldots \\
&\approx f(t) + \delta t\, \dot{f}(t)
\end{aligned}
$$

- Explicit Euler time integration

$$
f(t+\delta t) \leftarrow f(t) + \delta t\, \dot{f}(t)
$$

Brook Taylor
(1685-1731)

Leonhard Euler
(1707-1783)

Finite Difference Method

- Approximation of temporal derivative

$$f_t[i,j,t] \approx \frac{f[i,j,t+1] - f[i,j,t]}{\delta t}$$

Diffusion Flow

- Continuous PDE

$$\frac{\partial f(x,y,t)}{\partial t} = \lambda \left(\frac{\partial^2 f(x,y,t)}{\partial x^2} + \frac{\partial^2 f(x,y,t)}{\partial y^2} \right)$$

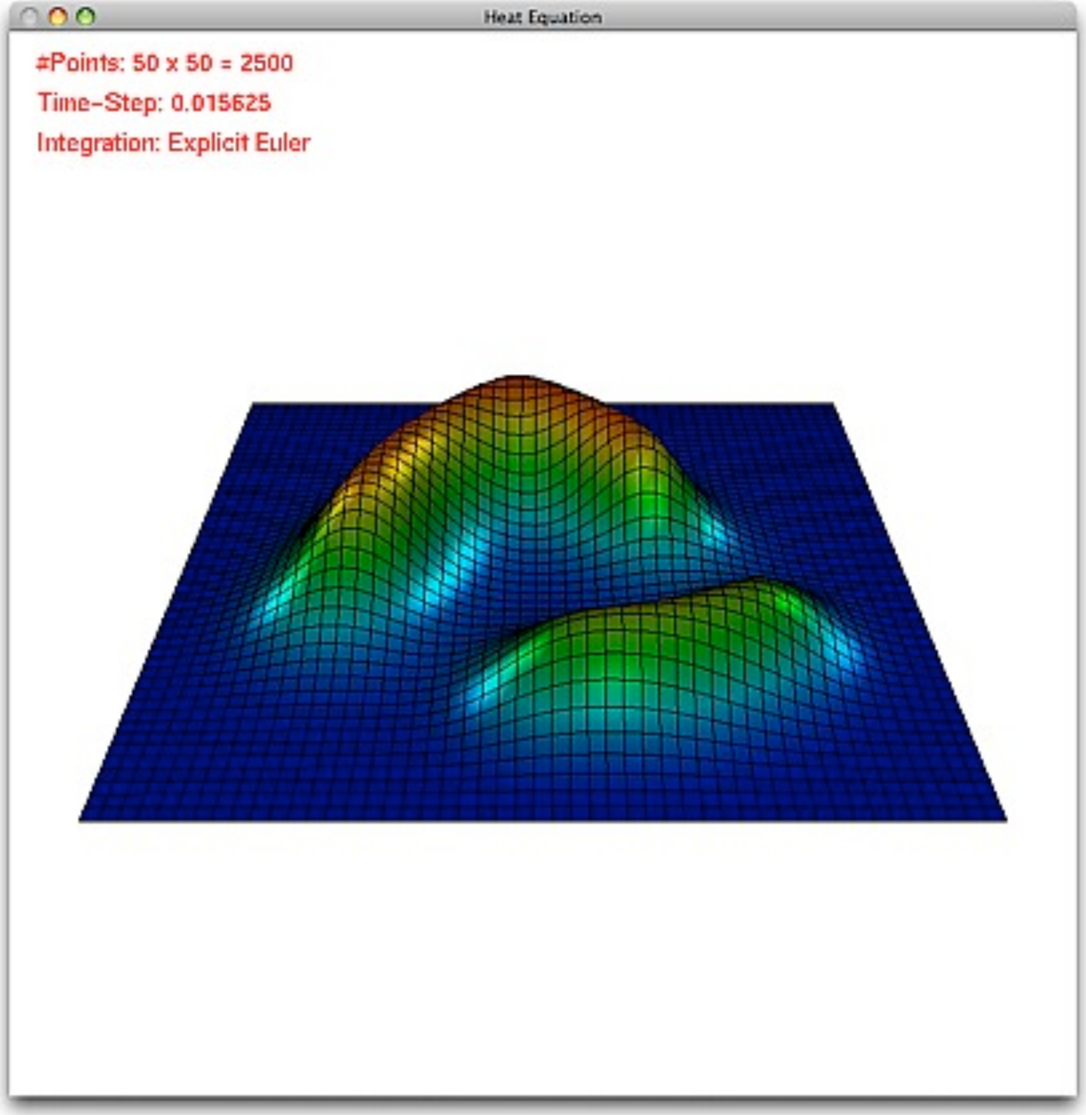
- Finite difference discretization

$$\frac{f[i,j,t+1] - f[i,j,t]}{\delta t} = \lambda \frac{f[i+1,j,t] + f[i-1,j,t] + f[i,j+1,t] + f[i,j-1,t] - 4f[i,j,t]}{h^2}$$

$$f[i,j,t+1] = f[i,j,t] + \delta t \lambda \frac{f[i+1,j,t] + f[i-1,j,t] + f[i,j+1,t] + f[i,j-1,t] - 4f[i,j,t]}{h^2}$$

$\delta t \lambda$ has to be small (<=1 in this case)

Diffusion in 2D

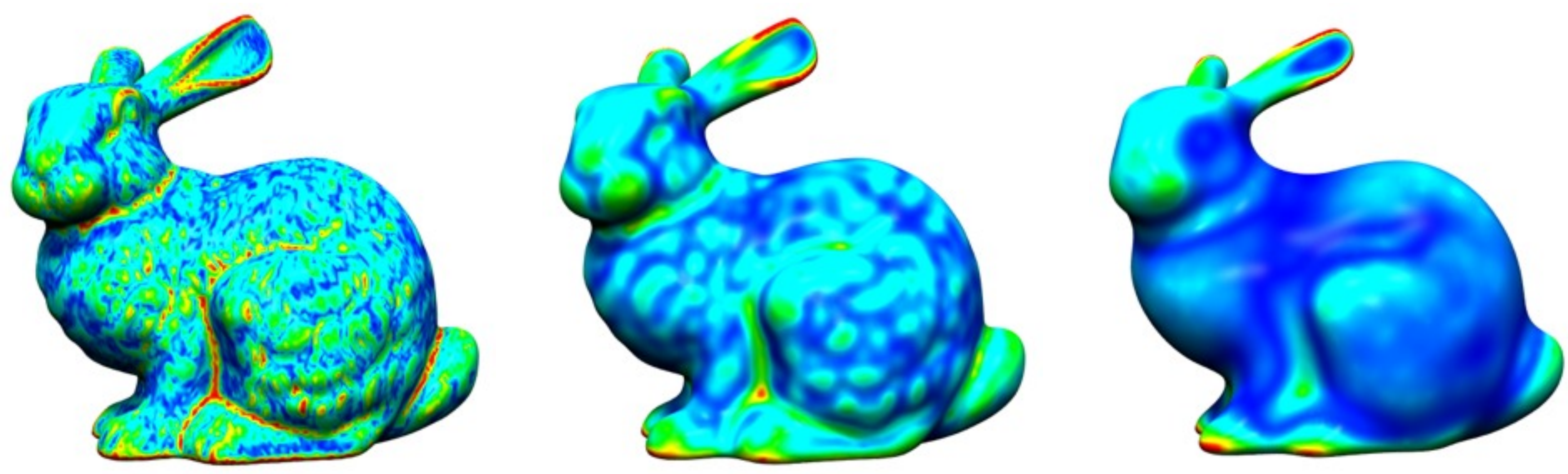


Demo

Diffusion Flow on Meshes

- Continuous $$\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta \mathbf{p}$$
- Discretization $$\mathbf{p}_i \leftarrow \mathbf{p}_i + \delta t \, \lambda \Delta \mathbf{p}_i$$

$\delta\lambda$ has to be small (~0.5 in exercises)



0 Iterations 10 Iterations 100 Iterations

Laplace Discretization

- Laplace discretization

$$\Delta \mathbf{p}_i = \frac{1}{\sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij}} \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} \left(\mathbf{p}_j - \mathbf{p}_i \right)$$

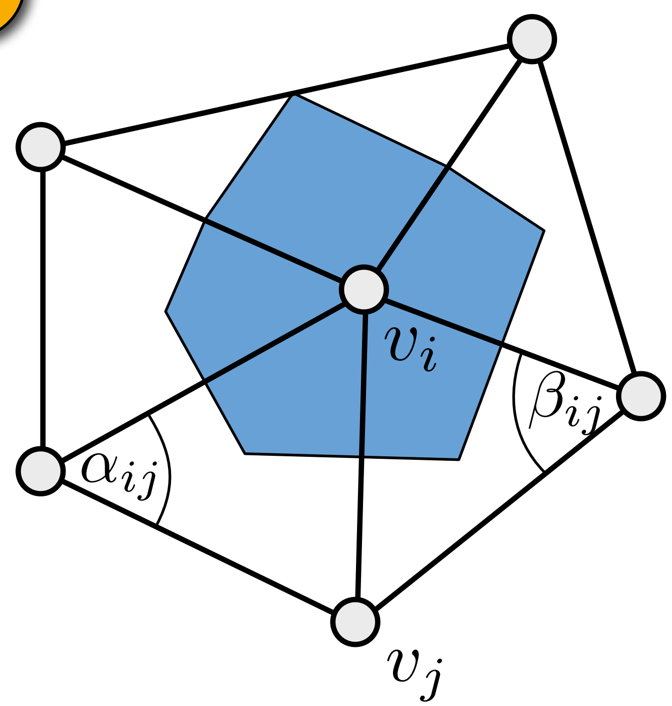
Note: No Voronoi area A_i for explicit smoothing

- Uniform Laplace

$$w_{ij} = 1$$

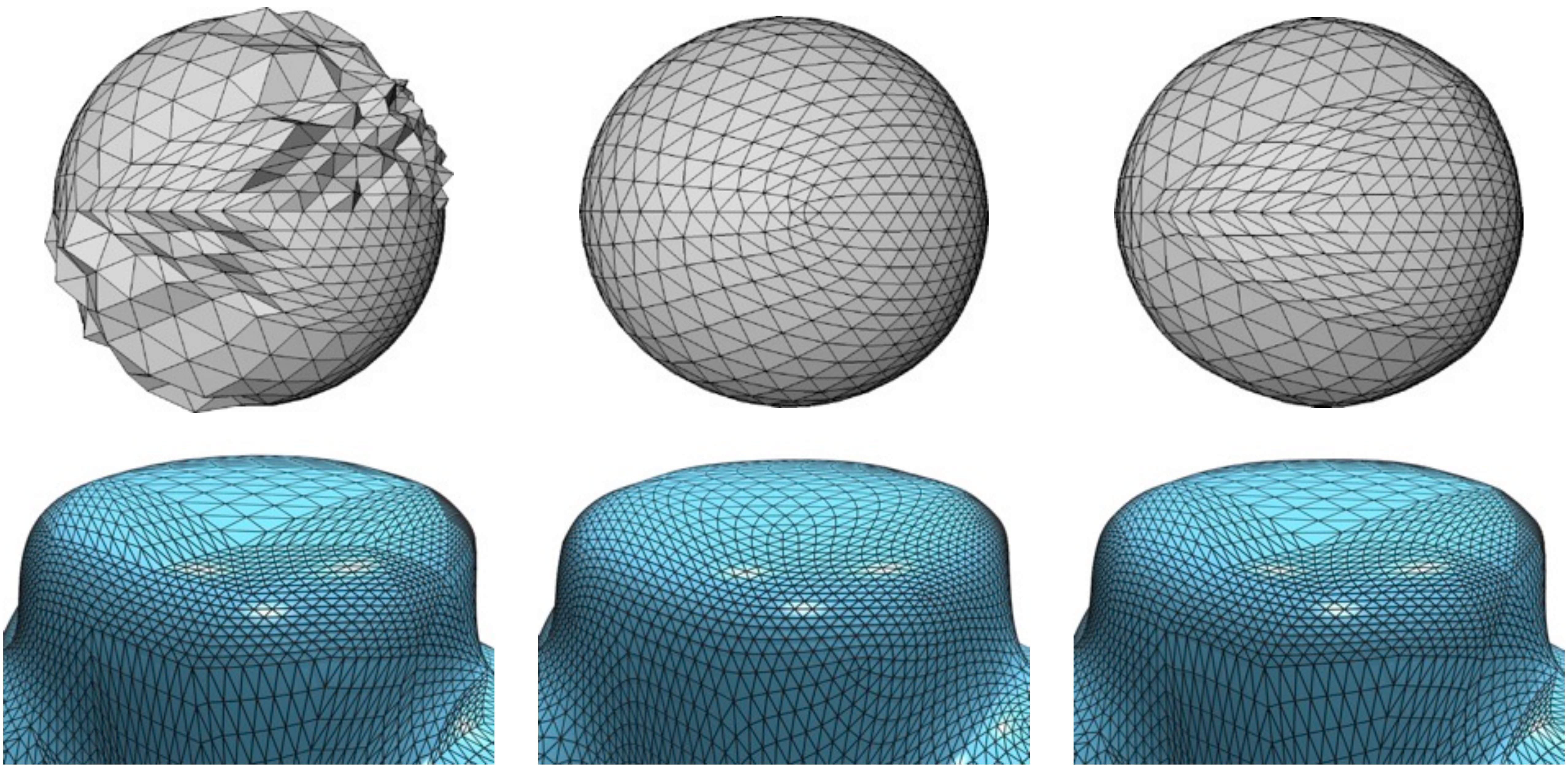
- Cotangent Laplace

$$w_{ij} = \cot \alpha_{ij} + \cot \beta_{ij}$$



Comparison

Original | Uniform Laplace | Cotan Laplace



Further Topics

- Implicit time integration

$$\mathbf{p}_i(t+1) = \mathbf{p}_i(t) + \delta t\, \lambda \Delta \mathbf{p}_i(t+1)$$

- Higher-order flows

$$\frac{\partial \mathbf{p}}{\partial t} \;=\; (-1)^{k+1} \lambda \Delta^k \mathbf{p}$$

- Boundary constraints
- Nonlinear flows, anisotropic flows

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Smoothing & **Fairing**

Fairness

- Idea: Penalize “unaesthetic behavior”

- Measure fairness
  - Principle of the simplest shape
  - Physical interpretation

- Minimize some fairness functional
  - Surface area, curvature
  - Membrane energy, thin plate energy

Nonlinear Energies

- Membrane energy (surface area)

$$\int_{\mathcal{S}} \mathrm{d}A \;\rightarrow\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}$$

- Thin-plate surface (curvature)

$$\int_{\mathcal{S}} \kappa_1^2 + \kappa_2^2 \,\mathrm{d}A \;\rightarrow\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}, \;\; \mathbf{n}(\delta\mathcal{S}) = \mathbf{d}$$

- Too complex... simplify energies

Membrane Surfaces

- Membrane energy (surface area)

$$\int_{\Omega} \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \,\mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

Variational Calculus in 1D

- 1D membrane energy

$$L(f) = \int_a^b {f'}^2(x)\ \mathrm{d}x \ \rightarrow \ \min$$

- Add test function u with $u(a) = u(b) = 0$

$$L(f + \lambda u) = \int_a^b (f' + \lambda u')^2 = \int_a^b {f'}^2 + 2\lambda f' u' + \lambda^2 {u'}^2$$

- If f minizes L, the following has to vanish

$$\left.\frac{\partial L(f + \lambda u)}{\partial \lambda}\right|_{\lambda=0} = \int_a^b 2 f' u' \ \stackrel{!}{=} \ 0$$

Variational Calculus in 1D

- Has to vanish for any u with $u(a) = u(b) = 0$

$$\int_a^b f'u' \;=\; \underbrace{[f'u]_a^b}_{=0} - \int_a^b f''u \;\stackrel{!}{=}\; 0 \quad \forall u$$

$$\int_0^1 f'g = [fg]_0^1 - \int_0^1 fg'$$

- Only possible if

$$f'' \;=\; \Delta f \;=\; 0$$

➡ *Euler-Lagrange equation*

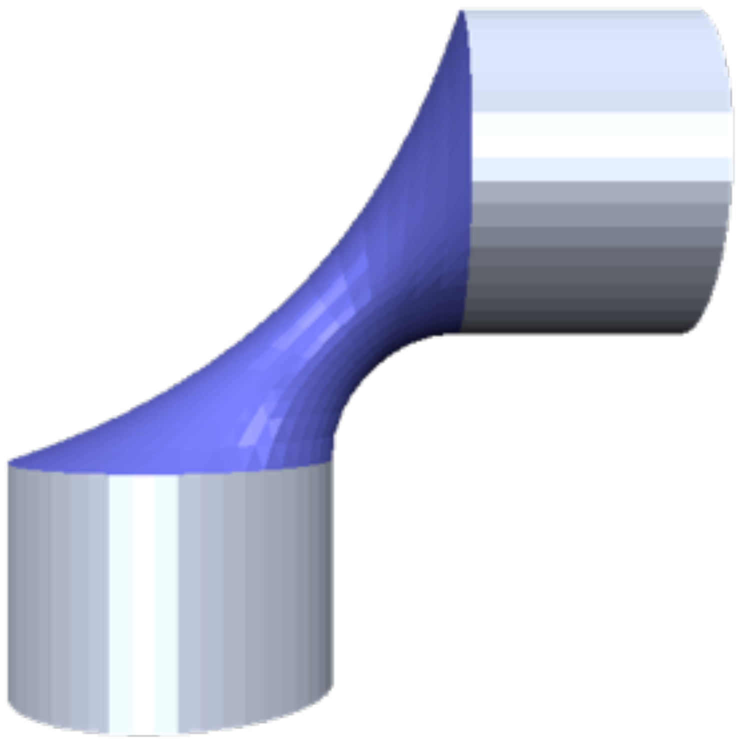
Membrane Surfaces

- Membrane energy (surface area)

$$\int_{\Omega} \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \, \mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

- Variational calculus

$$\Delta \mathbf{p} = 0$$



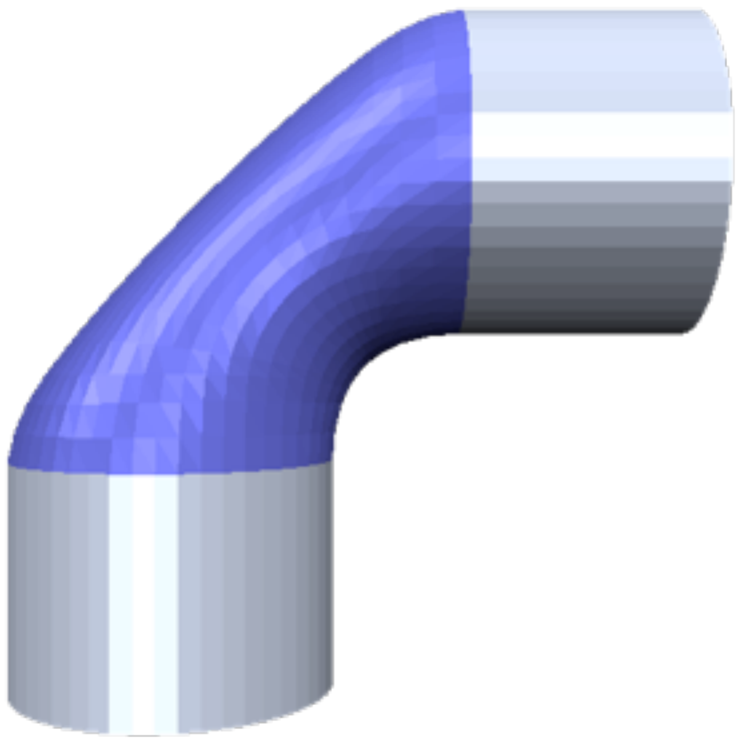
Thin-Plate Surfaces

- Thin-plate energy (curvature)

$$\int_{\Omega} \|\mathbf{p}_{uu}\|^2 + 2\|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \,\mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

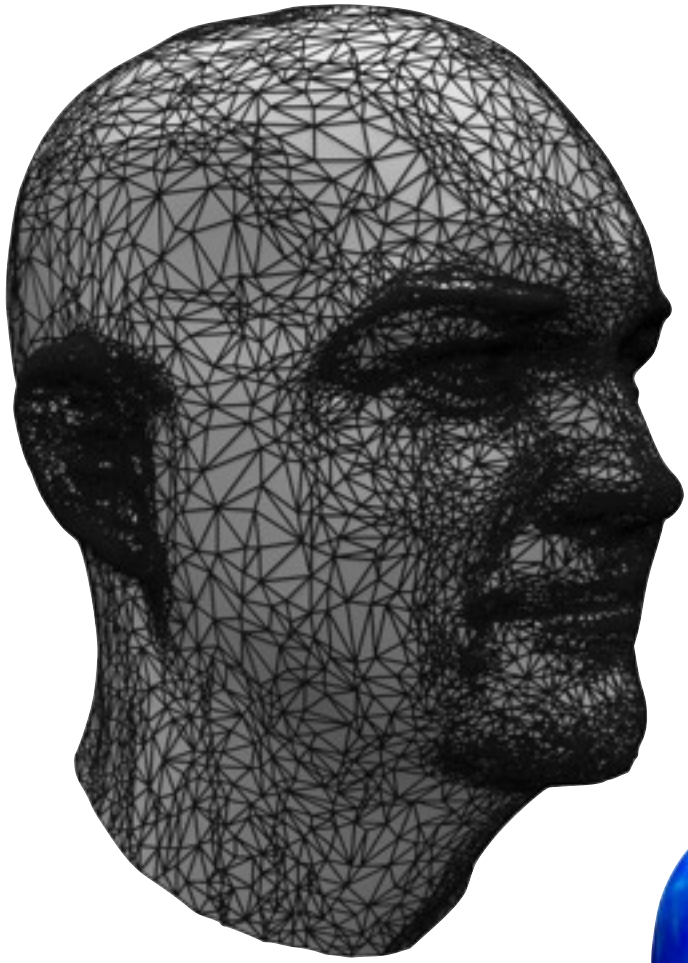
- Variational calculus

$$\Delta^2 \mathbf{p} = 0$$

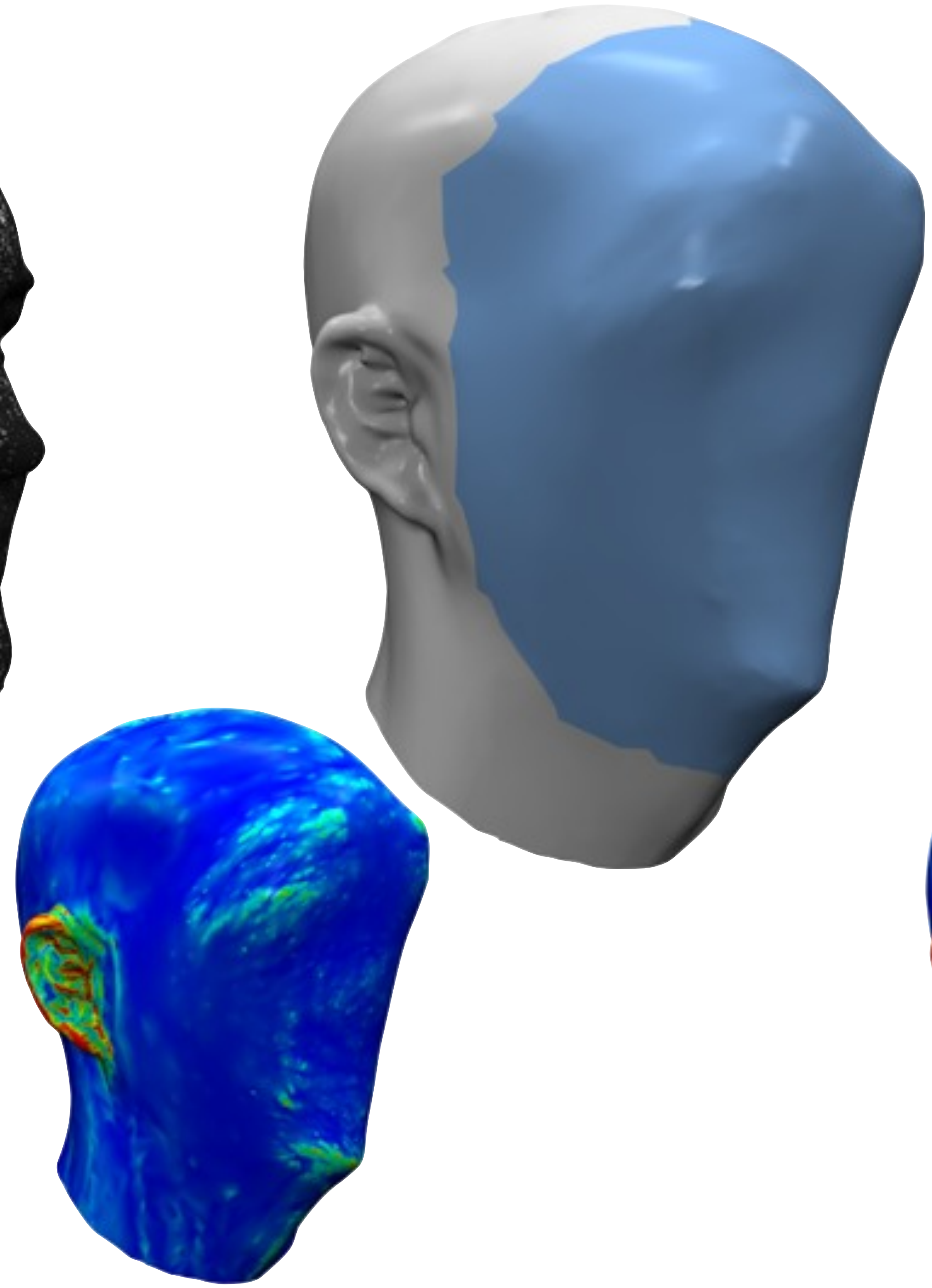


Laplace Discretization

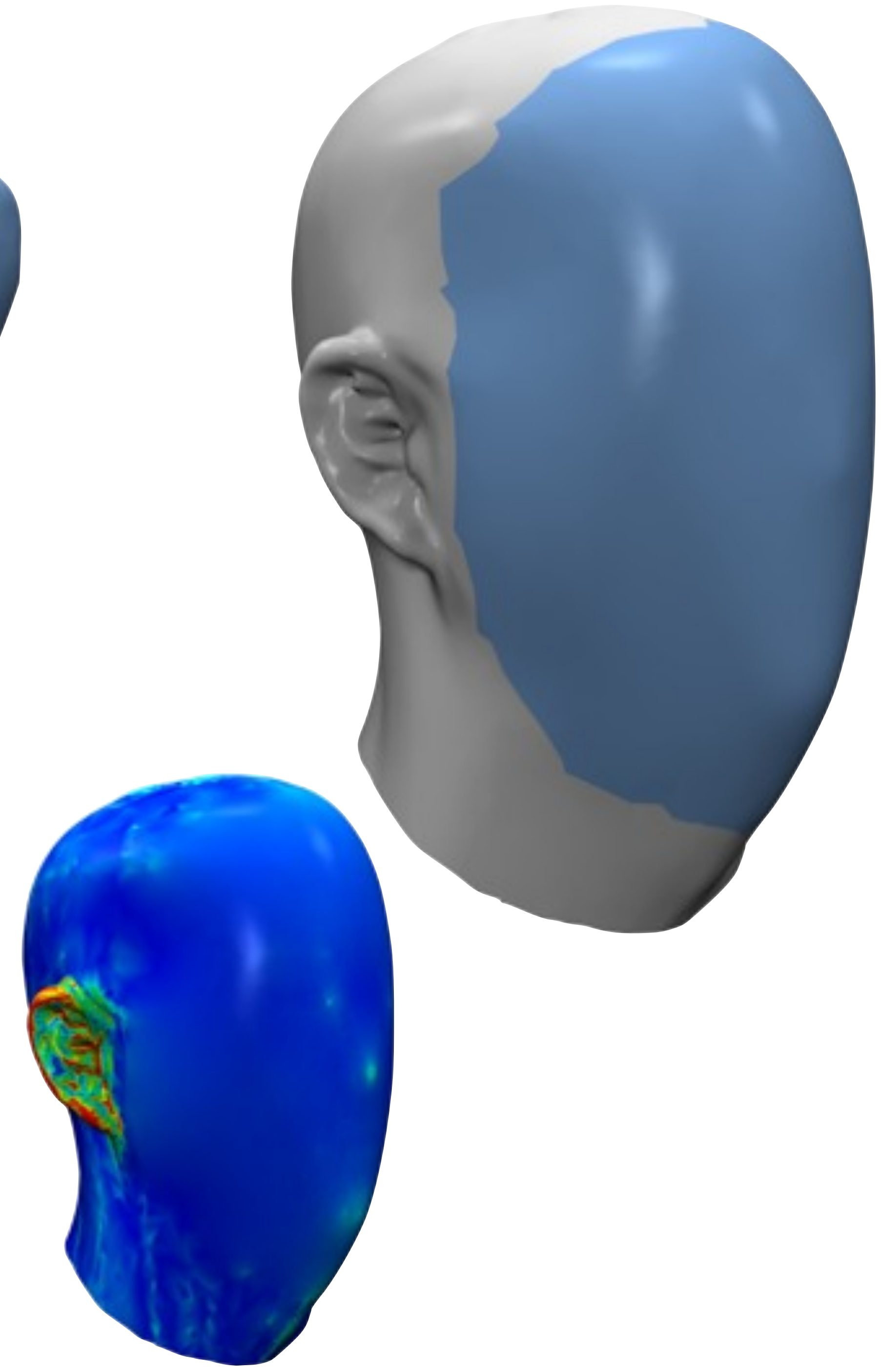
Original



Uniform Laplace



Cotan Laplace



Smoothing ⇔ Fairing

- Fair minimizer surfaces satisfy

$$\Delta^k \mathbf{p} = 0$$

- Hence they are stationary points of Laplacian flows

$$\frac{\partial \mathbf{p}}{\partial t} = (-1)^{k+1} \lambda \Delta^k \mathbf{p}$$

- Diffusion flow converges to fair surfaces

$$\int \left\| \frac{\partial^k \mathbf{p}}{\partial u^k} \right\|^2 + \left\| \frac{\partial^k \mathbf{p}}{\partial u^{k-1} \partial v} \right\|^2 + \left\| \frac{\partial^k \mathbf{p}}{\partial u^{k-2} \partial v^2} \right\|^2 + \cdots + \left\| \frac{\partial^k \mathbf{p}}{\partial v^k} \right\|^2 \mathrm{d}u\mathrm{d}v \rightarrow \min$$

- Explicit time integration corresponds to an iterative solution of the Euler-Lagrange equation

Literature

- Laplacian flow, curvature flow
  - The Book: Chapter 4
  - Taubin, *A Signal Processing Approach to Fair Surface Design,* SIGGRAPH 1995
  - Desbrun, Meyer, Schröder, Barr: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 1999

Literature

- Nonlinear smoothing
  - Bobenko & Schröder: *Discrete Willmore Flow*, SGP 2005
- Anisotropic smoothing
  - Bajaj & Xu: Anisotropic Diffusion of Surfaces and Functions of Surfaces, ACM Trans. on Graphics 22(1), 2003
  - Hildebrandt & Polthier, Anisotropic Filtering of Nonlinear Surface Feature, Eurographics 2004
- Bilaterial smoothing
  - Fleishman et al, Bilateral Mesh Denoising, SIGGRAPH 2003
  - Jones et al, Non-iterative Feature-Preserving Mesh Smoothing, SIGGRAPH 2003